The Extended True Count Theorem

Eric Farmer

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Reference:

[1] Thorp, Edward, *Does Basic Strategy Have The Same Expectation For Each Round?* Blackjack Forum **13.2** (1993), 27-44.

Introduction

In blackjack, the house advantage is the expected value of a hand played using basic strategy, and is typically expressed as a percentage of the initial wager. (Most casino games have an expected value of around -0.6%.) Basic strategy is that playing strategy which maximizes the expected value of the hand. This expected value is usually computed based on the assumption that a single player plays the hand starting with a full shoe. In reality, several players may be playing each hand, often with different strategies, and several hands are dealt from the shoe before reshuffling. The question arises whether the expected value of a particular hand played with a fixed strategy depends on these additional factors.

Card counting systems estimate the expected value of a hand using the "true count," which is a linear combination of the probabilities of each card value being dealt from the shoe. The True Count Theorem states that the expected true count does not depend on the number of hands already dealt.

Theorem 1 (Jalib) *The expected value of the true count after a card is revealed and removed from any deck composition is exactly the same as before the card was removed, for any balanced [or appropriately calculated unbalanced] count, provided you do not run out of cards.*

The Extended True Count Theorem is a generalization of this result, stating that the exact expected value (which is only estimated by the true count) for a fixed playing strategy depends on neither the number of hands already dealt nor the strategies of other players. This result is proved in [1]; the approach taken here is to make precise the class of functions of a deck composition for which the result applies, and to present the True Count Theorem as a special case.

Playing strategy and expected value

Some notation is required to establish the more general framework in which the Extended True Count Theorem applies. Let S be the set of cards in a full shoe, and let A be the set of all possible arrangements of subsets of cards in S:

$$A = \bigcup_{T \subseteq S} \Sigma(T)$$

Note that an element π in A may be viewed as an arrangement via a fixed canonical ordering of the elements of S. Given an arrangement π, let $|\pi|$ denote the number of cards in the arrangement. Two arrangements π and π' satisfy the relation $\pi \leq \pi'$ if π is a "prefix" of π'; that is, $|\pi| \leq |\pi'|$ and the first $|\pi|$ cards in π' consist of precisely those in π, in that order.

The "rules of the game" and playing strategy for a single player may be specified by the payoff function

$$w^*: A \to \Re$$

where $w^*(\pi)$ is the amount won by the player given that the cards in π are dealt, in that order, from the top of the shoe. There are two important things to note. First, playing strategy is implicit in the specification of w^*, since the outcome of the hand depends on which cards in π are dealt to the player (by hitting, splitting, etc.) and which are dealt to the dealer (by standing, busting, etc.). Indeed, there is nothing special about *where* the cards are dealt, or the particular rules of blackjack; all that is needed is a fixed strategy for removing cards from a shoe until it can be determined how much the player has won or lost. Second, not all of the cards in w^* may be necessary to make this determination. This "redundancy" in w^* is indicated in the following definition.

$$|w^*| = \min\{k : \pi, \pi' \in A, |\pi| \geq k, \pi \leq \pi' \Rightarrow w^*(\pi) = w^*(\pi')\}$$

Essentially, $w^*(\pi)$ depends on at most the first $|w^*|$ cards in π. That is, $|w^*|$ is the maximum number of cards that may be dealt in a single hand. The expected value of a hand may now be defined by the function

$$v_{w^*}: \wp(S) \to \Re$$

where

$$v_{w^*}(T) = \frac{1}{|\Sigma(T)|} \sum_{\pi \in \Sigma(T)} w^*(\pi)$$

Given a subset T of cards remaining in the shoe, each arrangement of cards being equally likely, the expected value of a hand dealt from that shoe is $v_{w^*}(T)$. The generalization of the true count is apparent here; in the case where $v_{w^*}(T)$ is a linear combination of card probabilities, $|w^*| = 1$.

Multi-player strategy and *k*-equivalence

Now consider a hand with several players, where player i uses the same fixed strategy as that specified by w^*. The outcome for player i may be specified by another payoff function w, which accounts for the additional cards dealt to the remaining players. These two payoff functions describe the same outcomes for player i in the following sense: given $k > 0$, payoff functions w and w^* are *k-equivalent* if for all $T \subseteq S$ such that $|T| \geq k$, there exists a bijection

$$\Psi : \Sigma(T) \to \Sigma(T)$$

such that for all π in $\Sigma(T)$, $w(\pi) = w^*(\Psi(\pi))$. That is, the outcomes specified by w^* for each of the possible arrangements of cards in T are simply permuted in w. The parameter k indicates the smallest number of cards in the shoe for which the two payoff functions yield the same distribution of outcomes; a convenient value for k is $\max\{|w|, |w^*|\}$.

Note that if two payoff functions w and w^* are k-equivalent, then for any subset T of at least k cards, the expected values of the payoff functions for T are the same; that is, $v_w(T) = v_{w^*}(T)$. To see that the single and multi-player payoff functions are k-equivalent (where $k = \max\{|w|, |w^*|\}$), suppose that in the multi-player case the cards for the dealer and player i are dealt from the top of the shoe, and all cards for the remaining players are dealt from the bottom. This effectively realizes the mapping Ψ in the above definition.

Card removal strategy and the Extended True Count Theorem

The Extended True Count Theorem deals with the expected value of the function v_w after some cards have been removed from the shoe. A (possibly non-deterministic) strategy for removing cards from the shoe may be specified by a function

$$\sigma : A \to [0,1]$$

which maps an arrangement of cards already dealt into a probability of dealing an additional card. Such a strategy is executed as follows: deal a single card with probability $\sigma(\langle\rangle)$. If card c is dealt, deal another with probability $\sigma(\langle c \rangle)$, etc.

Given a subset T of cards remaining in the shoe, each arrangement of cards being equally likely, a removal strategy σ induces a random variable $X_{T,\sigma}$

which is the resulting subset of cards still in the shoe after execution of the strategy. Note that for $X_{T,\sigma}$ to be well-defined, σ must have the following property:

$$(\forall \pi \in \Sigma(T))(\exists \pi' \in A)\pi' \leq \pi, \sigma(\pi') = 0$$

That is, σ must be guaranteed to stop before running out of cards in the shoe. Note also that the removal strategy need not correspond to the dealing of previous blackjack hands, let alone with fixed playing strategies. The Extended True Count Theorem applies to any removal strategy specified as above.

Theorem 2 (Extended True Count Theorem) *Given a set S of cards in a full shoe, each arrangement of cards being equally likely, let w* and w be the payoff functions for a player's fixed strategy while playing alone and with other players, respectively. If* σ *is a corresponding removal strategy, then*

$$E(v_w(X_{S,\sigma})) = v_w(S) = v_{w*}(S)$$

provided that

$$\min\left|X_{S,\sigma}\right| \geq \max\{|w|, |w*|\}$$

(that is, provided that "you do not run out of cards").

Proof. The latter equality follows from the k-equivalence of w and w^* explained in the previous section, where $k = \max\{|w|, |w^*|\}$. For the former equality, use induction on $|S| - \min|X_{S,\sigma}|$, or the maximum possible number of cards removed. If $|S| - \min|X_{S,\sigma}| = 0$, then no cards are removed. Otherwise,

$$E(v_w(X_{S,\sigma})) = (1 - \sigma(\langle\rangle))v_w(S) + \sigma(\langle\rangle)(\frac{1}{|S|}\sum_{c \in S} E(v_w(X_{S-\{c\},\sigma|_c})))$$

where $\sigma|_c$ is a restriction of the removal strategy given that card c has been dealt. More precisely,

$$\sigma|_c (\langle c_1, c_2, \ldots\rangle) = \sigma(\langle c, c_1, c_2, \ldots\rangle)$$

provided that card c does not appear twice in the resulting arrangement. By induction,

$$\begin{aligned}\frac{1}{|S|}\sum_{c\in S}E(v_w(X_{S-\{c\},\sigma|_c})) &= \frac{1}{|S|}\sum_{c\in S}v_w(S-\{c\}) \\ &= \frac{1}{|S|}\sum_{c\in S}\left(\frac{1}{|\Sigma(S-\{c\})|}\sum_{\pi\in\Sigma(S-\{c\})}w(\pi)\right) \\ &= \frac{1}{|\Sigma(S)|}\sum_{c\in S}\sum_{\pi\in\Sigma(S-\{c\})}w(\pi) \\ &= \frac{1}{|\Sigma(S)|}\sum_{\pi\in\Sigma(S)}w(\pi)\end{aligned}$$

The last equality follows from the requirement that

$$\min|X_{S,\sigma}| \geq |w|$$

Because of this, for any π in Σ(*S*), *w*(π) does not depend on all of the cards in π. Thus,

$$\begin{aligned}E(v_w(X_{S,\sigma})) &= (1-\sigma(\langle\rangle))v_w(S) + \sigma(\langle\rangle)\left(\frac{1}{|\Sigma(S)|}\sum_{\pi\in\Sigma(S)}w(\pi)\right) \\ &= (1-\sigma(\langle\rangle))v_w(S) + \sigma(\langle\rangle)v_w(S) \\ &= v_w(S)\end{aligned}$$