Exact Expected Values for Splitting Pairs in Blackjack

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Introduction

Computing the exact expected value of splitting, and re-splitting, a pair in blackjack generally requires significantly more time and/or memory than for the other player options, such as standing, hitting, or doubling down. The problem is complicated by the need to compute "single hand" expected values (i.e., conditioned on no further re-splits) for several different "shoe states," or compositions of card values remaining in the shoe. Closed form expressions are given for the expected number of split hands and the overall expected return from the split, in terms of the single hand expected values for the required shoe states. Recurrence relations are given for computing these expected values in terms of expectations that may be computed efficiently by removing some number of pair cards from the shoe. The results are presented so as to apply to both finite and infinite deck analysis, with minimal modifications depending on the desired model.

Definitions

In the following discussion, the *initial shoe state* is the "full shoe" with the two pair cards from the initial deal and the dealer's up card removed. The initial shoe state is represented by two parameters: p is the probability of a pair card being dealt from the remaining shoe, and s is the number of cards remaining in the shoe, with $s = 1$ by convention for an infinite deck. For example, consider splitting a pair of eights against a dealer six. For a single deck, p is 2/49 and s is 49; for an infinite deck, $p = 1/13$ and $s = 1$.

In addition, let $EV(X;a,b)$ be the expected value of drawing to a single pair card and playing the resulting hand, given the following conditions: if another pair card is drawn, it is played and not re-split; also, a pair cards and b non-pair cards have been removed from the initial shoe state. Similarly, $EV(N;a,b)$ is the expected value of drawing to a single pair card and playing the resulting hand, with the additional condition that a *non-pair* card is drawn first. Finally, $EV(P;a,b)$ is the expected value of drawing to a single pair card and playing the resulting hand, with the additional condition that a *pair* card is drawn first (i.e., another pair is drawn but not re-split). Note that for an infinite deck, these expected values are independent of a and b.

Finally, define the following auxiliary function:

$$f(x,n) = \begin{cases} (x)_n = x!/(x-n)! & \text{for finite decks} \\ x^n & \text{for infinite decks} \end{cases}$$

Probabilities and Expected Values

Starting with a given pair hand and corresponding initial shoe state, suppose that the player splits and re-splits at every opportunity, up to a maximum of n hands, and otherwise plays using a fixed strategy for every split hand. There are two possibilities. First, the player may split less than the maximum number of hands. The probability that the player splits exactly h hands (where $2 \leq h \leq n - 1$) is given by:

$$q(h) = \frac{1}{h}\binom{2h-2}{h-1}\frac{f(ps,h-2)\,f((1-p)s,h)}{f(s,2h-2)}$$

The ratio involving f is the probability of a particular arrangement of $h - 2$ pair cards and h non-pair cards being drawn to the initial pair. The remaining terms count the number of such valid arrangements, equal to the $(h - 1)^{st}$ Catalan number.

The second possibility is that the player splits the maximum number of hands, but completes (i.e., draws non-pair cards to) k of them (where $0 \leq k \leq n - 2$) before drawing additional pair cards to reach the maximum number of hands. The probability in this case is given by:

$$r(n,k) = \left(1 - \frac{k}{n-1}\right)\binom{n+k-2}{k}\frac{f(ps,n-2)\,f((1-p)s,k)}{f(s,n+k-2)}$$

Given these probabilities, the expected number of split hands and overall expected value of the split are, respectively:

E[number of hands] =

$$\sum_{h=2}^{n-1} q(h)\cdot h + \sum_{k=0}^{n-2} r(n,k)\cdot n$$

E[return] =

$$\sum_{h=2}^{n-1} q(h)\cdot h\cdot EV(N;h-2,h-1) +$$
$$\sum_{k=0}^{n-2} r(n,k)(k\cdot EV(N;n-2,k-1)+(n-k)EV(X;n-2,k))$$

In the special case of unlimited re-splits in an infinite deck, these expected values simplify to:

$$\text{E[number of hands]} = \sum_{h=2}^{\infty} q(h)\cdot h = \frac{2-2p}{1-2p}$$

$$\text{E[return]} = \frac{2-2p}{1-2p} EV(N)$$

Computing Expected Values for Finite Decks

For finite decks, the formula in the previous section gives the overall expected return from splitting a pair in terms of "single hand" expected values $EV(X;a,b)$ and $EV(N;a,b)$, conditioned on removing various numbers of pair cards and non-pair cards from the shoe. These may in turn be expressed via recurrence relations with base terms of the form $EV(X;a,0)$ and $EV(P;a,0)$ for various a, which are relatively straightforward to pre-compute simply by removing a additional pair cards from the shoe.

To do this, consider the following equations expressing $EV(X;a,b)$ and $EV(N;a,b)$ recursively according to whether an additional pair card or non-pair card is removed from the shoe:

$$EV(X;a,b) = \frac{ps-a}{s-a-b} EV(X;a+1,b) + \left(1 - \frac{ps-a}{s-a-b}\right) EV(X;a,b+1)$$

$$EV(N;a,b) = \frac{ps-a}{s-a-b-1} EV(N;a+1,b) + \left(1 - \frac{ps-a}{s-a-b-1}\right) EV(N;a,b+1)$$

Solving these equations for $EV(X;a,b+1)$ and $EV(N;a,b+1)$, respectively, yields recurrence relations involving strictly fewer numbers of *non-pair* cards removed from the shoe. Thus, the overall expected return may be computed recursively with base terms of the form $EV(X;a,0)$ and $EV(N;a,0)$.

Finally, $EV(N;a,0)$ may be computed in terms of $EV(X;a,0)$ and $EV(P;a,0)$ by solving the following equation:

$$EV(X;a,0) = \frac{ps-a}{s-a} EV(P;a,0) + \left(1 - \frac{ps-a}{s-a}\right) EV(N;a,0)$$